Riemann hypothesis from the Dedekind psi function

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Abstract. Let $\mathcal{P}$ be the set of all primes and $\psi(n) = n\prod_{n\in\mathcal{P},p|n}(1+1/p)$ be the Dedekind psi function. We show that the Riemann hypothesis is satisfied if and only if $f(n) = \psi(n)/n - e^{\gamma}\log\log n < 0$ for all integers $n > n_0 = 30$ (D), where $\gamma \approx 0.577$ is Euler's constant. This inequality is equivalent to Robin's inequality that is recovered from (D) by replacing $\psi(n)$ with the sum of divisor function $\sigma(n) \geq \psi(n)$ and the lower bound by $n_0 = 5040$. For a square free number, both arithmetical functions σ and ψ are the same. We also prove that any exception to (D) may only occur at a positive integer n satisfying $\psi(m)/m < \psi(n)/n$, for any $m < n$, hence at a primorial number N_n or at one its multiples smaller than N_{n+1} (Sloane sequence $A060735$). According to a Mertens theorem, all these candidate numbers are found to satisfy (D): this implies that the Riemann hypothesis is true.

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1. Introduction

Riemann zeta function $\zeta(s) = \sum_{n>1} n^{-s} = \prod_{p\in\mathcal{P}} \frac{1}{1-p^{-s}}$ (where the product is taken over the set $\mathcal{P}$ of all primes) converges for $\mathcal{R}(s) > 1$. It has a analytic continuation to the complex plane with a simple pole at $s = 1$. The Riemann hypothesis (RH) states that non-real zeros all lie on the critical line $\mathcal{R}(s) = \frac{1}{2}$. RH has equivalent formulations, many of them having to do with the distribution of prime numbers [1, 2].

Let $d(n)$ be the divisor function. There exists a remarkable parallel between the error term $\Delta(x) = \sum_{n\leq x} d(n) - x(\log x + 2\gamma - 1)$ in the summatory function of $d(n)$ (Dirichlet's divisor problem) and the corresponding mean-square estimates $|\zeta(\frac{1}{2}) + it|$ of $\zeta(s)$ on the critical line, see [3] for a review. This may explain Ramanujan's interest for highly composite numbers [4]. A highly composite number is a positive integer n such that for any integer $m < n$, $d(m) < d(n)$, i.e. it has more divisors than any positive integer smaller than itself.

This landmark work eventually led to Robin's formulation of RH in terms of the sum of divisor function $\sigma(n)$ [5, 6, 7]. More precisely,

$$\text{RH is true iff } g(n) = \frac{\sigma(n)}{n} - e^{\gamma}\log\log n < 0 \text{ for any } n > 5040. \tag{1}$$

The numbers that do not satisfy (1) are in the set $\mathcal{A} = \{2, 3, 4, 5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 30, 36, 48, 60, 72, 84, 120, 180, 240, 360, 720, 840, 2520, 5040\}$.

If RH is false, the smallest value of $n > 5040$ that violates the inequality should be a superabundant number [8, 9, 10], i.e. a positive number n satisfying

$$\frac{\sigma(m)}{m} < \frac{\sigma(n)}{n} \text{ for any } m < n. \tag{2}$$

No counterexample has been found so far.

In the present paper, Robin's criterion is refined by replacing the sum of divisor function by the Dedekind psi function $\psi(n) = \prod_{p \in \mathcal{P}, p|n}(1 + \frac{1}{p})$ ‡. Since $\psi(n) \leq \sigma(n)$, with equality when n is free of square, we establish the refined criterion

$$\text{RH is true iff } f(n) = \frac{\psi(n)}{n} - e^{\gamma} \log\log n < 0 \text{ for any } n > 30. \tag{3}$$

The numbers that do not satisfy (3) are in the set $\mathcal{B} = \{2, 3, 4, 5, 6, 8, 10, 12, 18, 30\}$.

A number that would possibly violate (3) should satisfy

$$\frac{\psi(m)}{m} < \frac{\psi(n)}{n} \text{ for any } m < n, \tag{4}$$

and be a primorial number $N_n = \prod_{i=1}^{n} p_i$ (the product of the first n primes) or one its multiples smaller than N_{n+1} (Sloane sequence $A060735$).

According to a Mertens theorem [2], one has $\lim_{n\to\infty} \frac{\psi(N_n)}{N_n} = \frac{e^{\gamma}}{\zeta(2)} log(p_n)$ and we show that none of the numbers larger than 30 in the sequence $A060735$ can violate (3). As a result, RH may only be true.

In the next section, we provide the proof of (3) and compare it with the Robin's criterion (1). Furthermore, we investigate the structure of numbers satisfying (4) and justify why they fail to provide counterexamples to RH.

Originally, Dedekind introduced $\psi(n)$ as the index of the congruence subgroup $\Gamma_0(n)$ in the modular group (see [11], p. 79). In our recent work, the Dedekind psi function plays a role for understanding the commutation relations of quantum observables within the discrete Heisenberg/Pauli group [12]. In particular, it counts the cardinality of the projective line $\mathbb{P}_1(\mathbb{Z}_n)$ of the lattice $\mathbb{Z}_n \times \mathbb{Z}_n$. The relevance of the Dedekind psi function $\psi(n)$ in the context of RH is novel. For other works aiming at a refinement of Robin's inequality, we mention [13], [7] and [14].

2. A proof of the refined Robin's inequality

Let us compute the sequence $\mathcal{S}$ of all positive numbers satisfying (4). For $2 < n < 10^5$, one obtains $S = \{N_1 = 2, 4, N_2 = 6, 12, 18, 24, N_3 = 30, 60, 90, 120, 150, 180, N_4 = 210, 420, 630, 840, 1050, 1260, 1470, 1680, 1890, 2100, N_5 = 2310, 4620, 6930, 9240\}$, which are the first terms of Sloane sequence $A060735$, consisting of the primorials N_n and their multiples up to the next primorial N_{n+1}.

It is straightforward to check that about half of the numbers in S are not superabundant (compare to Sloane sequence $A004394$).

‡ The Dedekind function $\psi(n)$ should not be confused with the second Chebyshev function $\psi_T(n) = \sum_{p^k \leq n} \log p$.

Based on calculations performed on the numbers in the finite sequence S, we are led to three properties that the infinite sequence $A060735$ should satisfy

Proposition 1: For any $l > 1$ such that $N_n < lN_n < N_{n+1}$ one has $f(lN_n) < f(N_n)$.

Proof: One may use $\tilde{f}(n) = \frac{\psi(n)}{n \log\log n}$ instead of $f(n)$ to simplify the proof.

When l is prime, one observes that $lN_n = p_1 p_2 \cdots l^2 \cdots p_n$ for some $p_j = l$. The corresponding Dedekind psi function is evaluated as

$$\psi(lN_n) = (p_j^2 + p_j) \prod_{i \neq j} \psi(p_i) = l\psi(N_n).$$

Then, with

$$\tilde{f}(lN_n) = \frac{\psi(lN_n)}{lN_n \log\log(lN_n)} = \frac{\psi(N_n)}{N_n \log\log(lN_n)},$$

one concludes that

$$\frac{\tilde{f}(lN_n)}{\tilde{f}(N_n)} = \frac{\log\log N_n}{\log\log(lN_n)} < 1 \text{ in the required range } 1 < l < p_{n+1}.$$

When l is not prime, a similar calculation is performed by decomposing l into a product of primes and by using the multiplicative property of $\psi(n)$.

Proposition 2: Given $l \geq 1$, for any m such that $lN_n < m < (l+1)N_n < N_{n+1}$ one has $f(m) < f(N_n)$.

Proof: This proposition is proved by using inequality (4) at $n = (l+1)N_n$

$$\frac{\psi(m)}{m} < \frac{\psi[(l+1)N_n]}{(l+1)N_n} \text{ for any } m < (l+1)N_n$$

and the relation $\psi[(l+1)N_n] = (l+1)\psi(N_n)$. As a result

$$\tilde{f}(m) < \frac{\psi(N_n)}{N_n \log\log m} = \frac{\log\log N_n}{\log\log m} \tilde{f}(N_n) < \tilde{f}(N_n) \text{ since } m > N_n.$$

Proposition 3: There exist *infinitely many* prime numbers p_n such that $f(N_{n+1}) > f(N_n)$.

Remarks on the proposition 3: Based on experimental evidence in table 1, one would expect that $f(N_{n+1}) < f(N_n)$ and that the proposition 3 is false.

Similarly, one would expect that $\theta(p_n) < p_n$ for any n. For instance it is known [15] that

$$\theta(n) < n \text{ for } 0 < n \leq 10^{11}.$$

Many oscillating functions were studied in the context of the prime number theorem. It was believed in the past that, for any real number x, the function $\Delta_1(x) = \pi(x) - \mathrm{li}(x)$ (where li(x) is the logarithmic integral) is always negative. However, J E. Littlewood has shown that $\Delta_1(x)$ changes sign infinitely often at some large values $x > x_0$ [16]. The

n	10	10^3	10^5	10^7
$\frac{\theta(p_n)}{p_n}$	0.779	0.986	0.99905	0.999958
$\frac{\tilde{f}(N_{n+1})}{\tilde{f}(N_n)}$	0.987	0.9999980	0.99999999921	0.99999999999975
$\frac{k_n \log k_n}{p_{n+1}\log p_{n+1}}$	0.938	1.00378	1.000447	1.0000423

Table 1. An excerpt of values of $\theta(p_n)/p_n$ and $\frac{\tilde{f}(N_{n+1})}{\tilde{f}(N_n)}$ versus the number of primes in the primorial N_n.

smallest value x_0 such that for the first time $\pi(x_0) > \mathrm{li}(x_0)$ holds is called the Skewes number. The lowest present day value of the Skewes number is around 10^{316}.

In what concerns the function $\Delta_4(x) = \theta(x) - x$, according to theorem 1 in [17] (see also [18], Lemme 10.1), there exists a positive constant c_4 such that for sufficiently large T, the number of sign changes of $\Delta_4(x)$ in the interval $[2, T]$ is

$$V_4(T) \geq c_4 \log T.$$

Justification of proposition 3

According to theorem 34 in [19] (also used in [18])

$$\theta(x) - x = \Omega_{\pm}(x^{1/2} \log_3 x) \text{ when } x \to \infty,$$

where $\log_3 x = \log\log\log x$. The omega notation means that there exist infinitely many real numbers x satisfying

$$\theta(x) \geq x + \frac{1}{2}\sqrt{x}\log_3 x = k_x, \tag{5}$$

$$\text{and } \theta(x) \leq x - \frac{1}{2}\sqrt{x}\log_3 x$$

If $x = p_n$, for some n then

$$\theta(p_n) \geq p_n + \frac{1}{2}\sqrt{p_n}\log_3 p_n = k_n. \tag{6}$$

Otherwise, let us denote p_n the first prime number preceeding x, one has

$$\theta(p_n) = \theta(x) \geq k_x \geq k_n,$$

that is similar to (6).

At a primorial $n = N_n$, $\psi(N_n) = \prod_{i=1}^{n}(1 + p_i)$ so that $\psi(N_{n+1}) = (1 + p_{n+1})\psi(p_n)$. One would like to show that there are infinitely many prime numbers $p_n > 2$ such that

$$\frac{\tilde{f}(N_{n+1})}{\tilde{f}(N_n)} = (1 + \frac{1}{p_{n+1}})\frac{\log\theta(p_n)}{\log\theta(p_{n+1})} = \frac{1 + \frac{1}{p_{n+1}}}{1 + \log(1 + \frac{\log p_{n+1}}{\theta(p_n)})/\log\theta(p_n)} > 1.$$

By contradiction, let us assume that the reverse inequality holds for those prime numbers $x = p_n$ satisfying (6)

$$\frac{\log\theta(p_n)}{p_{n+1}} < \log(1 + \frac{\log p_{n+1}}{\theta(p_n)})$$

$$\text{with } \theta(p_n) \geq p_n + \frac{1}{2}\sqrt{p}_n \log_3 p_n. \tag{7}$$

Taking the development of the logarithm in the first equation of (7) one obtains $\frac{\log k_{p_n}}{p_{n+1}} < \frac{\log p_{n+1}}{k_{p_n}}$, that is

$$k_{p_n} \log k_{p_n} < p_{n+1} \log p_{n+1} \text{ for } k_{p_n} = p_n + \frac{1}{2}\sqrt{p}_n \log_3 p_n. \tag{8}$$

The inequality (8) contradicts the calculations performed in table 1 for $10 < n < 10^7$. We conclude that our proposition 3 is correct in a finite range of p_n's. In addition, since there are infinitely many prime numbers satisfying (6), proposition 3 is also satisfied for a infinite range of values.

The Riemann hypothesis

Propositions 1 and 2 show that the worst case for the inequality (3), if not satisfied, should occur at a primorial N_n. Proposition 3 deals about the distribution of values of $f(N_n)$ at large n. Propositions 1 and 2 are needed in the proof of RH, based on the refined Robin inequality.

Let us first show that RH $\Rightarrow$ (3).

This is easy because if RH is true, then Robin's inequality (1) is true, for any $m > 5040$, including at primorials $m = N_n$, $m = N_{n+1}$ and so on [5], despite the result established in proposition 3 that there are infinitely many values of n such that $f(N_{n+1}) > f(N_n)$ §. Since N_n is free of square, one has $\psi(N_n) = \sigma(N_n)$ so that the refined inequality (3) is satisfied at any $m = N_n$. This means that if RH is true then, according to proposition 1, (3) is satisfied at lN_n, where $N_n < lN_n < N_{n+1}$ and, according to proposition 2, it is also satisfied at any m between consecutive values lN_n and $(l+1)N_n$ of the sequence $A060735$.

The reverse implication (3) $\Rightarrow$ RH is similar to that for the Robin's inequality (theorem 2 in [5]). We observe that there exists an infinity of numbers n such that $\tilde{g}(n) = \frac{\sigma(n)}{n \log\log n} > e^\gamma$ and the bound on $\frac{\psi(n)}{n}$ is such that

$$\text{for } n \geq 3,\ \frac{\psi(n)}{n} \leq \frac{\sigma(n)}{n} \leq e^\gamma \log\log n + \frac{0.6482}{\log\log n}.$$

To prove that RH is true, it is sufficient to prove that no exception to the refined Robin's criterion may be found.

At large value of primorials N_n, we use Mertens theorem about the density of primes $\log p_n \prod_{k=1}^{n}(1 - \frac{1}{p_k}) \sim e^{-\gamma}$, or the equivalent relation ‖

$$\frac{1}{\log p_n}\frac{\psi(N_n)}{N_n} \equiv \frac{1}{\log p_n}\prod_{k=1}^{n}(1 + \frac{1}{p_k}) \sim \frac{e^\gamma}{\zeta(2)}, \tag{9}$$

§ In the first version of this paper, it was expected that for any p_n, one would have $\theta(p_n) < p_n$ and as result $f(N_{n+1}) < f(N_n)$. But this property is unnecessary for showing that the refined Robin inequality is equivalent to RH.

‖ A better approximation could be obtained from proposition 9 in [20], i.e. from $\prod_{p \leq x}(1 + \frac{1}{p}) = \frac{e^\gamma}{\zeta(2)} \log x + O(1/\log x)$.

and the lower bound given p. 206 of [5]

$$\text{for } p_n \geq 20000, \;\; \log\log N_n > \log p_n - \frac{0.123}{\log p_n}. \tag{10}$$

Using (9) and (10), and with $e^{\gamma}(\zeta(2)^{-1} - 1) \approx -0.698$, one obtains a lower bound

$$\text{for } p_n \geq 20000, \;\; f(N_n) < -0.698 \log p_n + \frac{0.220}{\log p_n} \sim -6.89. \tag{11}$$

For values of $2 \leq n \leq 100000$, computer calculations can be performed. The calculation of $f(N_n) = g(N_n) < 0$ is fast using the multiplicative property of $\sigma(n)$, i.e. using $\sigma(N_n) = \prod_{i=1}^{n}(1 + p_i)$. One find a decreasing function $g(N_n)$ that is negative if $n > 3$, i.e. $N_n > 30$, as expected (and in agreement with the exceptions in the sets $\mathcal{A}$ and $\mathcal{B}$).

n	3	10	10^2	10^3	10^4	10^5
$f(N_n) = g(N_n)$	0.22	-1.67	-4.24	-6.23	-8.06	-9.83

Table 2. The approximate value of the function $f(N_n) = g(N_n)$ versus the number of primes in the primorial N_n. The smallest primorial in the table is $N_n = 30$ and the highest one is $N_{100000} \approx 1.9 \times 10^{563920}$.

According to the calculations illustated in the table II and the bound established in (11), there are no exceptions to the refined Robin's criterion. Since Robin's criterion has been shown to be equivalent to RH hypothesis, RH may only be true.

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